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FIELD OF LONG INTERNAL LEE WAVES IN A PLANE-PARALLEL SHEAR LAYER

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A linear formulation is used to examine a three-dimensional problem on long steady-state internal waves formed by the movement of a plane-parallel shear flow over a short (relative to the depth of the water) isolated seamount. In contrast to [1, 2] (where a general linear formulation was used in a study of a field of internal lee waves in a uniform flow of an exponentially stratified fluid) and [3, 4] (where an asymptotic analysis was made of forced waves in a stably stratified flow with a velocity shift), in the present study we use a quasistatic approximation to obtain a series of double integrals representing an exact solution to the given problem for shear flow and arbitrary stable stratification of the fluid. The solution is obtained in elementary functions for a mountain of model form. Examples are presented of calculation of the near region of a field of internal lee waves in uniform and shear flows for an empirical Weisshall–Brent frequency profile.

1. Let a flow of an ideal, incompressible, stably stratified fluid of constant depth H travel from infinity with the velocity U(z) to an isolated underwater obstacle $z = -H + hf(x, y)$. Meanwhile, $\max|f| = 1$, $h \ll H$, $f \to 0$ for $x^2 + y^2 \to \infty$; x and y are the horizontal coordinates; z is the vertical coordinate. The x axis is directed along the incoming flow, while the z axis is directed vertically upward. The origin of the coordinate system coincides with the undisturbed free surface.

In a quasistatic approximation, the steady-state wave field created by the obstacle in the the flow is described by the equations

$$\begin{gathered} Uu_x + wU_z = -\rho_0^{-1}p_x,\ Uv_x = -\rho_0^{-1}p_y, \\ p_z = -\rho g,\ U\rho_x + w\rho_{0z} = 0,\ u_x + v_y + w_z = 0 \end{gathered} \tag{1.1}$$

with the boundary conditions

$$p = \rho_0 g\zeta,\ U\zeta_x = w\ (z = 0),\ w = hUf_x\ (z = -H), \tag{1.2}$$

where u, v, and w are components of the vector of the wave velocities; p and ρ are perturbations of pressure and density; ζ is the displacement of the free surface; and $\rho_0(z)$ is the undisturbed density profile. The subscripts denote differentiation with respect to the corresponding coordinate. Along with (1.2), we need to satisfy the radiation condition. The latter consists of the fact that all of the principal wave disturbances are concentrated downstream (x > 0).

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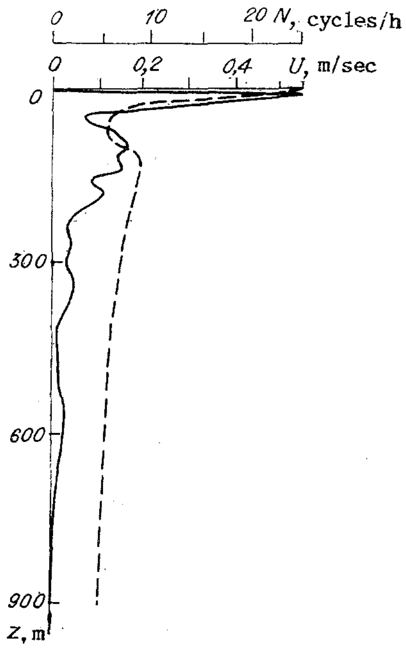

Fig. 1

We will assume that the Miles condition [5]

$$\mathrm{Ri}\,(z) = N^2(z)/U_z^2(z) > 0{,}25 \quad (-H < z < 0).$$

is satisfied, this condition establishing the absence of unstable natural vibrations.

After we reduce system (1.2) to a single equation relative to w and we resort to Fourier transformation with respect to the variables x and y (with the parameters μ and ν), we obtain the following boundary-value problem to determine the Fourier transformations W of the vertical component of velocity:

$$(\rho_1\psi_z)_z + \rho_1 N_1^2\lambda^2\psi = 0 \quad (-H < z < 0); \tag{1.3}$$

$$\psi_z - g\lambda^2\psi = 0 \ (z = 0), \ \psi = ik\cos\theta\cdot h\bar{f} \ (z = -H). \tag{1.4}$$

Here, $\psi = W/U$; $\rho_1 = \rho_0 U^2$; $N_1^2 = N^2c^2/U^2$; $\lambda^2 = (c\cos\theta)^{-2}$; $c = U(0)$; $N^2 = -g\rho_{0z}/\rho_0$ is the square of the Weisshall-Brent frequency; $k^2 = \mu^2 + \nu^2$; $\mu = k\cos\theta$; $\nu = k\sin\theta$; $\bar{f}(k, \theta)$ is the transformation of the Fourier function f(x, y). We will assume that $U(z) \neq 0$, $z \in [-H, 0]$.

Let $\Phi(z, \lambda^2)$ be the solution of Eq. (1.3) satisfying the boundary condition at z = 0:

$$\Phi_z(0, \lambda^2) - g\lambda^2\Phi(0, \lambda^2) = 0$$

We will seek the solution of problem (1.3-1.4) in the form $\psi = A\Phi$ (where A is a constant). Having inserted this expression into the second condition of (1.4), we find A and, thus,

$$\psi = ik\cos\theta\cdot h\bar{f}M(\lambda^2, z), \quad M(\lambda^2, z) = \Phi(z, \lambda^2)/\Phi(-H, \lambda^2). \tag{1.5}$$

Using the relation $U\zeta_x = w$ and the inversion formulas, we obtain the following from (1.5) for the vertical displacements of the fluid particles

$$\zeta(x, y, z) = h(2\pi)^{-1}\int_L\int_0^\infty \bar{f}M(\lambda^2, z)\,k\exp\{ikR\cos(\theta - \gamma)\}\,dk\,d\theta, \tag{1.6}$$

where integration over θ is performed from 0 to 2π, with circumvention of the poles of the integrand in accordance with the radiation condition: $R^2 = x^2 + y^2$; $x = R\cos\gamma$; $y = R\sin\gamma$.

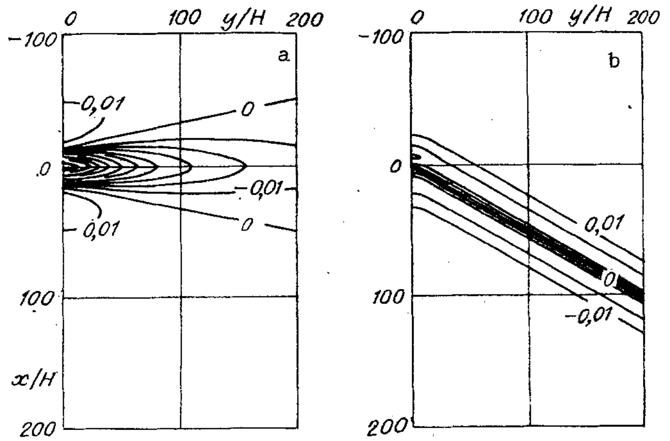

Fig. 2

2. The function $\Phi(z, \lambda^2)$ is an integral function of the parameter λ^2, which means that the function $M(\lambda^2, z)$ is meromorphic. Its poles are eigenvalues of the Sturm-Liouville problem corresponding to (1.3-1.4). Let

$$\lambda_n^2, \psi_n(z) \left(n=1, 2, 3, \ldots; \lambda_1^2 < \lambda_2^2 < \ldots; \|\psi_n\|^2 = \rho_1 g \psi_n^2(0) + \int_{-H}^{0} \rho_1 N_1^2 \psi_n^2 dz = 1 \right)$$

be the set of all eigenvalues and orthonormalized eigenfunctions. Using the theorem on the expansion of meromorphic functions into partial fractions, we find that $M(\lambda^2, z)$ can be represented by a uniformly convergent series [6]:

$$M(\lambda^2, z) = \varphi_0(z) - \rho_1(-H) \sum_{n=1}^{\infty} \varphi_n(z) \{(\lambda^2 - \lambda_n^2)^{-1} + \lambda_n^{-2}\},$$
$$\varphi_0(z) = \Phi(z, 0)/\Phi(-H, 0), \; \varphi_n(z) = \psi_n(z) \psi_{n_z}(-H). \tag{2.1}$$

With allowance for (2.1), we change (1.6) to the form

$$\zeta(x, y, z) = h\left\{I_0 + \sum_{n=1}^{\infty} \alpha_n (I_{n1} + I_{n2})\right\}, \; I_0 = \varphi_0(z) f(x, y),$$
$$I_{n1} \quad \mathrm{Re}\, \pi^{-1} \int_L \int_0^{\infty} \bar{f} \cos^2\theta \cdot k \left(\mathrm{fr}^{-2} - n^2 \cos^2\theta\right)^{-1} \exp\{ikR\cos(\theta - \gamma)\}\, dk\, d\theta, \tag{2.2}$$
$$I_{n2} = n^{-2} f(x, y), \; \alpha_n = -\rho_1(-H) \varphi_n(z) n^2 \lambda_n^{-2}, \; \mathrm{fr} = c\lambda_n/n.$$

Equation (2.2) represents the exact solution of the problem in the form of a series in the modes of the internal waves. Then expanding $\varphi_0(z)$ into a generalized Fourier series in a system of eigenfunctions $\{\psi_n\}$ and using the well-known properties of the Sturm-Liouville problem [7], we can show that

$$\varphi_0(z) = \rho_1(-H) \sum_{n=1}^{\infty} \varphi_n(z) \lambda_n^{-2} \; (-H < z \leqslant 0). \tag{2.3}$$

With allowance for (2.1) and (2.3), we find from (2.2) that

$$\zeta(x, y, z) = \begin{cases} h \sum_{n=1}^{\infty} \alpha_n I_{n1}, & z \neq -H \\ h f(x, y), & z = -H \end{cases}$$

It follows from this that the terms I_0 and I_{n2} (n = 1, 2, 3, ...) in (2.2) describe local effects in the immediate vicinity of the obstacle. These effects, connected with flow around the obstacle, depend weakly on the stratification and the velocity shift. The terms I_{n1}(n = 1, 2, 3 ...) represent the field of forced internal waves downstream.

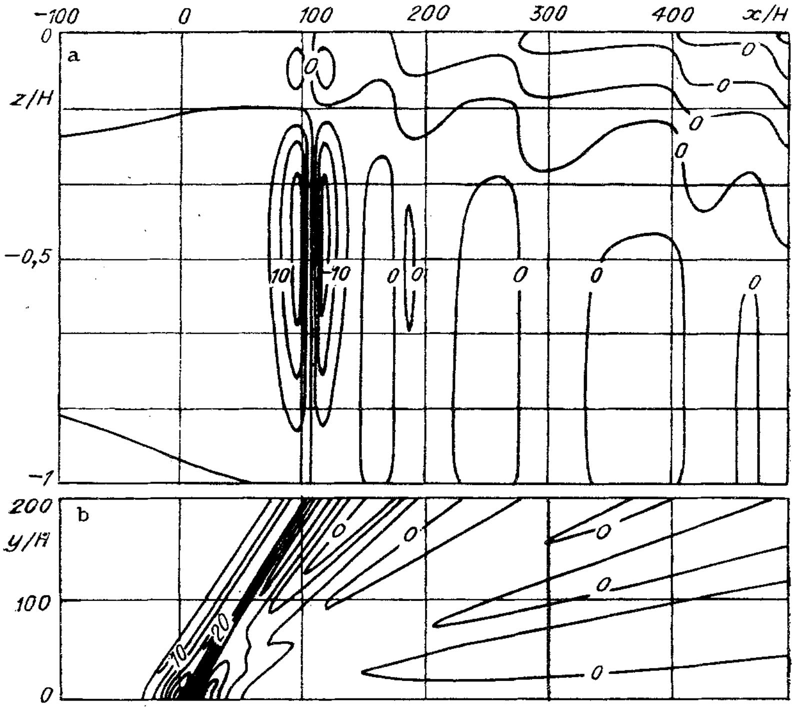

Fig. 3

Using the perturbation method and the condition of a "solid cover" on the surface, we obtain an explicit expression for I_0

$$I_0 = \left\{1 - \int_{-H}^{z} \rho_1^{-1}(\xi)\, d\xi \Big/ \int_{-H}^{0} \rho_1^{-1}(\xi)\, d\xi\right\} f(x, y).$$

3. In numerical calculations, the obstacle was modeled by the function

$$f(x, y) = [(x/A)^2 + (y/B)^2 + 1]^{-3/2},$$
$$\bar{f}(k, \theta) = AB \exp\,[-k\varkappa(\theta)],\ \varkappa(\theta) = (A^2 \cos^2\theta + B^2 \sin^2\theta)^{1/2}.$$

In this case, the integral of k in (2.2) is calculated analytically and expression (2.2) reduces to the form

$$\zeta(x, y, z) = h\left\{I_0 + \sum_{n=1}^{\infty} \alpha_n \left(\operatorname{Re} AB\pi^{-1} \int_L \cos^2\theta \cdot \Delta^{-1}(\theta)\, d\theta + I_{n2}\right)\right\}, \qquad (3.1)$$
$$\Delta(\theta) = (\mathrm{fr}^{-2} - n^2 \cos^2\theta)\,[\varkappa(\theta) - iR\cos(\theta - \gamma)]^2.$$

Integration in (3.1) is performed from $-\pi/2$ to $\pi/2$, with circumvention of the poles of the integrand along a small semicircle lying below the poles at Re < 0 and above them at Re > 0. Analogous to [8], the remaining integral in (3.1) can be calculated by means of the residue theorem and the solution of the problem can be calculated in elementary functions. The cumbersome final expression for the displacements of the fluid particles is omitted here due to space limitations.

The eigenvalues λ_n^2 and eigenfunctions $\psi_n(z)$ were found by numerical solution of the spectral problem for assigned distributions N(z) and U(z). We took the empirical profile N(z) represented in Fig. 1 by the solid line as the model of the distribution of the Weishall-Brent frequency. Ocean depth H in the measurement region was 900 m. The distribution of flow velocity in the depth direction (dashed line) was given by the model distribution with allowance for the principal properties of shear flows.

The characteristic horizontal scales of the underwater obstacle were chosen from the condition A, B ≥ 10H, which justifies the use of the longwave approximation [9] and is con-

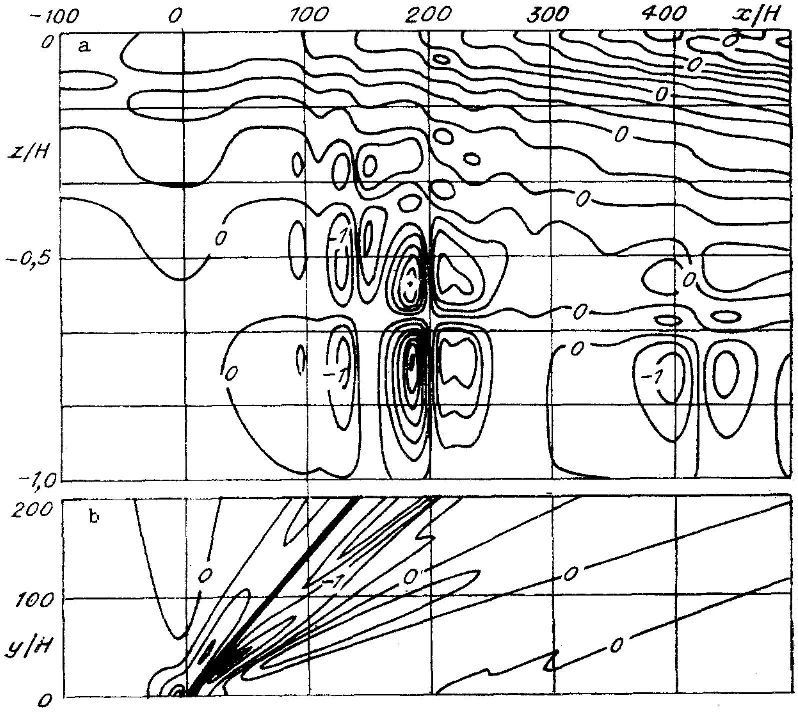

Fig. 4

sistent with actual conditions. The values taken for the parameters A, B, and h in the calculations were A/H = 15, B/H = 15, and h/H = 0.1.

To determine the effect of a velocity shift on the characteristics of lee waves in the immediate vicinity of an underwater obstacle, we performed calculations with and without allowance for the velocity shift, other conditions being equal. The velocity of nonshear flow was taken equal to U(0).

It was found that the main laws governing the formation of the field of forced waves in shear flow are the same as in a nonshear flow [2, 9]. Wave disturbances created by an individual internal mode with the number n are localized in the neighborhood of vertical planes making the angle $\pm\gamma_n$ with the xOz plane if $fr_n = c\lambda_n > 1$ (supercritical mode). The value of γ_n is determined by the formula $\gamma_n = \arcsin(fr_n^{-1})$. If $fr_n < 1$ (subcritical mode), then the contribution of this mode is significant only in the neighborhood of the underwater obstacle and decreases rapidly with increasing distance from it. Figure 2 shows the horizontal structure of an individual internal mode. The results shown from calculations of the sixth [subcritical, $fr_6 = 0.97$ (a)] and seventh [supercritical, $fr_7 = 1.12$ (b)] modes for the above model parameters are accurate to within the multiplier $h\alpha_n(z)$. Due to symmetry, the figure shows only half of the wave pattern. The largest contribution to the overall wave field is made by the first supercritical mode (i.e., the mode with the number ℓ such that $fr_{\ell-1} < 1 < fr_\ell$). The value of γ_ℓ determines the width of the region associated with the main wave perturbations. It should be noted that the values $fr_n = 1$ in the given model are resonance values. Estimates of the amplitudes of the waves for the critical flow velocities cannot be obtained within the framework of the linear long-wave approximation. In this case, a special analysis is needed with allowance for the effects of nonlinearity and dispersion – as has been done, for example, for waves in a uniform fluid [10].

The main differences between the wave fields in the shear and uniform flows are seen in the overall wave pattern. Figures 3 and 4 show vertical [at the distance y/H = 200 (a)] and horizontal [at the depth z/H = −0.45 (b)] sections of a field of lee waves without and with an allowance for the velocity shift, respectively. Due to symmetry, only half the wave pattern is shown on the horizontal sections. Summation in (3.1) was continued until the following condition was satisfied in the theoretical region

$$\max_{x,y} |\zeta_n(x,y) - \zeta_{n-1}(x,y)| \Big/ \max_{x,y} |\zeta_{n-1}(x,y)| < 0{,}01,$$

where ζ_n is the value of ζ calculated from (3.1) with allowance for n terms in the sum. It follows from the comparative analysis that allowance for the velocity shift leads to a change in the width of the region corresponding to the main wave distortions, a decrease in the amplitude of the internal waves, and a significant change in the vertical structure of the lee wave field. The higher modes make a larger contribution to the overall wave field in a shear flow than in a nonshear flow. This last finding is consistent with the suggestion, made in [11] after analysis of data from observations in the equatorial zone of the Indian Ocean, that higher modes may be dominant in the overall wave motion.

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